ON THE ABC CONJECTURE AND DIOPHANTINE APPROXIMATION BY RATIONAL POINTS

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ABSTRACT. We show that an earlier conjecture of the author, on diophantine approximation of rational points on varieties, implies the "abc conjecture" of Masser and Oesterlé. In fact, a weak form of the former conjecture is sufficient, involving an extra hypothesis that the variety and divisor admit a faithful group action of a certain type. Analogues of this weaker conjecture are proved in the split function field case of characteristic zero, and in the case of holomorphic curves (Nevanlinna theory).

The proof of the latter involves a geometric generalization of the classical lemma on the logarithmic derivative, due to McQuillan. This lemma may be of independent interest.

This paper discusses some conjectures that, if true, would imply the following conjecture, known as the Masser-Oesterlé "abc conjecture."

Conjecture 0.1 (Masser-Oesterlé). Let $\epsilon > 0$. Then there is a constant C, depending only on ϵ , such that for all triples $a,b,c \in \mathbb{Z}$ with a+b+c=0 and (a,b,c)=1, the following inequality holds:

(0.1.1)
$$\max\{|a|,|b|,|c|\} \le C \prod_{p|abc} p^{1+\epsilon}.$$

It has been known for some time that this conjecture would follow from other well-known conjectures; for example, see ([Vo 1], §5.ABC). In particular, it would be a consequence of the following conjecture:

Conjecture 0.2 ([Vo 1], Conjecture 5.2.6). Let k be a number field, let S be a finite set of places of k containing all the archimedean places of k, let X be a smooth projective curve over k, let D be an effective divisor on X without multiple

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points, let \mathscr{K} denote the canonical line sheaf on X, let \mathscr{A} be an ample line sheaf on X, let $\epsilon > 0$, and let r be a positive integer. Then the inequality

$$m_{S,k}(D,P) + h_{\mathcal{K},k}(P) \le \frac{\log |D_{k(P)/\mathbb{Q}}|}{[k(P):k]} + \epsilon h_{\mathcal{A},k}(P) + O(1)$$

holds for all $P \in X(\overline{\mathbb{Q}})$ with $P \notin \operatorname{Supp} D$ and $[k(P) : k] \leq r$.

Here $h_{\mathcal{K},k}$ and $h_{\mathcal{A},k}$ denote logarithmic heights normalized relative to k, and $m_{S,k}$ is the proximity function for D. See Section 1 for details.

Although it is stated here only for curves, this conjecture is still out of reach at the present time, due to the problem of dealing with the discriminant term.

The purpose of the present paper is to show how the abc conjecture would follow from the following, possibly weaker, conjecture.

Conjecture A ([Vo 1], Conjecture 3.4.3). Let k be a number field, let S be a finite set of places of k containing all the archimedean places of k, let X be a smooth complete variety over k, let D be a normal crossings divisor on X (assumed effective and without multiple components), let $\mathscr K$ denote the canonical line sheaf on X, let $\mathscr A$ be a big line sheaf on X, and let $\epsilon > 0$. Then there exists a proper Zariski-closed subset Z of X, depending only on X, D, $\mathscr A$, and ϵ , such that the inequality

$$m_{S,k}(D,P) + h_{\mathcal{K},k}(P) \le \epsilon h_{\mathcal{A},k}(P) + O(1)$$

holds for all $P \in (X \setminus (Z \cup \operatorname{Supp} D))(k)$.

This conjecture has the obvious advantage of dealing only with rational points, at the expense of allowing X to have arbitrary dimension. In fact, to get arbitrarily small values of ϵ , Conjecture A would need to be known for certain pairs (X,D) of arbitrarily large dimension.

On the other hand, the pairs (X,D) are special in the sense that they admit a faithful group action of $\mathbb{G}_{\mathrm{m}}^{\dim X-1}$. Therefore, it would be sufficient to prove the following weakening of Conjecture A:

Conjecture B. Conjecture A holds under the additional assumption that there is a semiabelian variety G, of dimension $\dim X - 1$, acting faithfully on X in such a way that the action preserves D.

If all points in X had finite stabilizers under the group action, then a quotient X/G might exist, and would be a curve. In that case Conjecture B may possibly follow from the fact that Conjecture A is known for curves. Unfortunately, there always exist points in X with infinite stabilizers, so this approach does not work. However, even without forming a quotient, it is possible to prove analogues of Conjecture B in the split function field case of characteristic zero and in the case of holomorphic curves $f: \mathbb{C} \to X$ (Nevanlinna theory).

The proof of the latter involves a result of McQuillan. An immediate corollary of this result, Corollary 5.2, gives a Nevanlinna-like inequality involving pull-backs of holomorphic differential forms on the domain space. This corollary corresponds to the trivial fact for algebraic maps, that if $f: C \to X$ is an algebraic map from a complex projective curve C to a projective variety X and ω is a meromorphic differential form on X such that $f^*\omega \neq 0$, then the number of zeroes minus the number of poles of $f^*\omega$ must equal 2g(C) - 2. This corollary may be useful for translating theorems about curves on varieties into corresponding theorems on holomorphic curves.

The proof of McQuillan's result mentioned above involves a geometric generalization of the classical lemma on the logarithmic derivative, Theorem A.2, which may also be of interest. An earlier geometric version of the logarithmic derivative lemma was proved by Noguchi ([N], Lemma 2.3). His version was stated in terms of a (nonzero) global section of $\Omega_X^1(\log D)$; it follows from the present version. Others have generalized Noguchi's work, but McQuillan was the first to remove the dependence on global sections.

This paper is organized as follows. Section 1 sets notation and recalls some of the basic definitions. Section 2 gives a characterization of the exceptional set Z in Conjecture B. Section 3, which forms the heart of the paper, contains the proof that Conjecture B implies the abc conjecture.

Sections 4 and 5 prove analogues of Conjecture B in the split function field case and in the Nevanlinna theory case, respectively. Section 6 introduces a hierarchy of variations of Conjecture A and similar diophantine statements. For example, it has been known for decades that it is often productive to prove a statement in the function field case before trying to prove it for number fields. The hierarchy contains this, as well as the corresponding observation about Nevanlinna theory. In light of this hierarchy, Sections 4 and 5 represent the progression of Conjecture B through the steps in the hierarchy.

Section 7 of the paper gives another way of obtaining a weak form of the abc conjecture (for $\epsilon > 26$) from Conjecture A. This particular variant works with one particular three-fold, with D=0. It also gives an example of how Conjecture B on a certain rational projective surface would imply something abc-like.

Finally, the paper concludes with an appendix giving a proof of Proposition 5.1, since a proof has not appeared in print. This proof follows the ideas of McQuillan [McQ 2], where the D=0 case is proved, and of P.-M. Wong [W]. See also [McQ 3].

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§1. Notation and definitions

In this paper, a **variety** is an integral scheme, separated and of finite type over a field. The conventions regarding $\mathbb{P}(\mathscr{E})$ and $\mathscr{O}(1)$ on $\mathbb{P}(\mathscr{E})$, where \mathscr{E} is a vector sheaf, are as in EGA. In other words, a point on $\mathbb{P}(\mathscr{E})$ corresponds to a hyperplane in the corresponding fiber of \mathscr{E} .

Unless otherwise specified, k will denote a global field of characteristic zero, M_k its set of places, and S a finite subset of M_k containing all archimedean places. Each

place $v \in M_k$ has an associated almost-absolute value $\|\cdot\|_v$, normalized as follows. If k is a number field with ring of integers R, then

$$||x||_v = \begin{cases} |\sigma(x)|^{[k_v:\mathbb{R}]} & \text{if } v \text{ is archimedean, corresponding to } \sigma, \bar{\sigma} \colon k \hookrightarrow \mathbb{C}; \\ (R:\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(x)} & \text{if } v \text{ is non-archimedean, corresponding to} \\ & \text{a prime ideal } (0) \neq \mathfrak{p} \subseteq R. \end{cases}$$

If v is a complex place, then $\|\cdot\|_v$ fails to satisfy the triangle inequality, hence the term almost-absolute value.

If k is a function field, then we assume without loss of generality that its field of constants is algebraically closed, and define $||x||_v = \exp(-\operatorname{ord}_v(x))$ for $x \in k^{\times}$. (Of course, the abc conjecture itself is already known in the function field case, but some parts of this paper require the notation in the function field case.)

With this choice of normalization, the product formula $\prod_{v \in M_k} ||x||_v = 1$, $x \in k^{\times}$, holds without multiplicities.

If $P \in \mathbb{P}^n(k)$ is a rational point with homogeneous coordinates $[x_0 : \cdots : x_n]$, then we define the **height**

$$h_k(P) = \sum_{v \in M_k} \log \max\{\|x_0\|_v, \dots, \|x_n\|_v\}.$$

(We shall always assume that homogeneous coordinates lie in the field of definition of the point.) If E is a finite extension of k (with compatible global field structure), then $h_E(P) = [E:k]h_k(P)$, so we may define $h_k(P)$ for $P \in \mathbb{P}^n(\bar{k})$ by $h_k(P) = h_E(P)/[E:k]$ for any field E containing k(P).

The well-known functoriality and additivity properties of Weil heights then allow us to define a height $h_{\mathcal{L},k}\colon X(\bar{k})\to\mathbb{R}$ for any complete variety X over k and any line sheaf \mathcal{L} on X. These heights are only defined up to O(1).

For places $v \in M_k$ and Cartier divisors D on a complete variety X, let

$$\lambda_{D,v} \colon (X \setminus \operatorname{Supp} D)(\bar{k}_v) \to \mathbb{R}$$

be Weil functions, normalized so that if D=(f) locally then $\lambda_{D,v}$ behaves like $-\log ||f||_v$ near Supp D. Then the height function may be decomposed as a sum

$$(1.1) h_{\mathscr{O}(D),k}(X) = \sum_{v \in M_k} \lambda_{D,v}(P) + O(1)$$

for all $P \in X(k) \setminus \operatorname{Supp} D$. See ([L], Chap. 10) for details on Weil functions.

For a finite subset S of M_k , Weil functions allow us to break the height $h_{\mathcal{O}(D),k}$ up into two parts, the **proximity function**

$$m_{S,k}(D,P) := \sum_{v \in S} \lambda_{D,v}(P)$$

and the **counting function**

$$N_{S,k}(D,P) := \sum_{v \notin S} \lambda_{D,v}(P) ,$$

so that

$$h_{\mathcal{O}(D),k}(P) = m_{S,k}(D,P) + N_{S,k}(D,P) + O(1)$$

for all $P \in X(k) \setminus \text{Supp } D$, by (1.1). As with the height, the proximity and counting functions satisfy

$$m_{T,E}(D,P) = [E:k]m_{S,k}(D,P)$$
 and $N_{T,E}(D,P) = [E:k]N_{S,k}(D,P)$,

where T is the set of places in M_E lying over places in S. This allows us to define $m_{S,k}$ and $N_{S,k}$ for $P \in X(\bar{k}) \setminus \text{Supp } D$ so that (1.2) still holds. We also note that the proximity and counting functions are additive and functorial in D.

§2. The exceptional set in Conjecture B

This section characterizes the exceptional set Z in Conjecture B in terms of the group action.

We start with some facts about groups acting on varieties. Let G be a group variety over a field k of characteristic zero, and let X be a smooth variety over k on which G acts. We have a morphism

$$\alpha: G \times X \to X \times X$$

whose first component is the map $G \times X \to X$ defining the group action and whose second component is the projection to X. Viewing both as schemes over X via the second projection, the morphism α becomes a morphism over X. This induces a morphism of relative tangent bundles:

$$\alpha_* : T_{G \times X/X} \to \alpha^* T_{X \times X/X}$$
.

Pulling back the above map by the map $(0, \mathrm{Id}_X) \colon X \to G \times X$ and using the isomorphisms $T_G \cong \mathscr{O}_G^{\dim G}$ and $T_{X \times X/X} \cong \mathrm{pr}_1^* T_X$ gives a map

$$\mathscr{O}_X^{\dim G} \to (0, \mathrm{Id}_X)^* \alpha^* \mathrm{pr}_1^* T_X \cong T_X$$
.

Taking $\wedge^{\dim G}$ of this map then gives a map

$$\mathscr{O}_X \to \wedge^{\dim G} T_X$$
,

which is equivalent to giving a section

$$\omega \in \Gamma(X, \wedge^{\dim G} T_X)$$
.

Lemma 2.1. The above section ω is nonzero if and only if the kernel of the group action has dimension zero.

Proof. Suppose first that the kernel H has nonzero dimension. Since H is normal, G/H exists, and α factors through $\pi\colon G\times X\to (G/H)\times X$. Therefore α_* factors through $\pi_*\colon T_{G\times X/X}\to \pi^*T_{(G/H)\times X/X}$. Since $\wedge^{\dim G}T_{(G/H)}=0$, the section ω is zero

Conversely, suppose that $\dim H = 0$. Let η be the generic point of X, and let $L = K(\eta)$. The map $G_{\overline{L}} \to X_{\overline{L}}$ defined by $g \mapsto g\eta$ has finite fibers, hence is étale. This implies that α_* is an isomorphism at $(0, \eta)$, so ω is nonzero at η .

- **Definition 2.2.** Let G be a group variety acting on a variety X. An orbit Gx (for $x \in X$) is **degenerate** if $\dim Gx < \dim G$.
- **Lemma 2.3.** Let G be a commutative group variety over a field k of characteristic zero, and let X be a smooth variety over k on which G acts faithfully. Then there exists an open dense G-invariant subset $U \subseteq X$, a variety V over k, and a morphism $\pi \colon U \to V$ exhibiting V as a quotient U/G.

Proof. Let U_0 be the open set on which the above-defined section ω is nonzero. Since G is commutative, α is G-equivariant (if we let G act on $G \times X$ by acting on the second factor, and on $X \times X$ by acting on both factors); hence ω is G-invariant, so U_0 is also G-invariant. Since G acts faithfully, Lemma 2.1 implies that $\omega \neq 0$, so $U_0 \neq \emptyset$.

Moreover, Lemma 2.1 applied to each orbit in U_0 implies that the stabilizer of every point in U_0 is finite. Therefore ([**SGA 3**], V 8.1) applies, giving an open dense subset $U \subseteq U_0$ and a quotient morphism $\pi: U \to U/G$.

Definition 2.4. Let X be a scheme, let $P \in X$ be a regular point, and let D be a divisor on X. Then D has **normal crossings** at P if, locally in the étale topology, D is effective and can be written as a principal divisor $D = (x_1 \cdots x_r)$, where x_1, \ldots, x_r are elements of the maximal ideal $\mathfrak{m}_{P'}$ in the local ring at the point P' in the étale neighborhood, and the images of x_1, \ldots, x_r in $\mathfrak{m}_{P'}/\mathfrak{m}_{P'}^2$ are linearly independent over the residue field at P'. We say that D is a **normal crossings divisor** on a regular scheme X if it has normal crossings at all $P \in X$.

Note that, under this definition, a normal crossings divisor must be effective and reduced. The notion of normal crossings does not make sense at a singular point.

Remark 2.5. Definition 2.4 can be restated as follows. A divisor D has normal crossings at P if and only if there exists an étale neighborhood $\pi\colon X'\to X$ of P and functions $x_1,\ldots,x_r\in \mathscr{O}(X')$, such that (i) $\pi^*D=(x_1\cdots x_r)$, and (ii) for each subset $I\subseteq\{1,\ldots,r\}$, the subscheme cut out by the ideal generated by $\{x_i:i\in I\}$ is regular of codimension #I.

Definition 2.6. Let D be a divisor on a scheme X. Then we say that the pair (X, D) is **regular** at a point $P \in X$ if X is regular at P and D has normal crossings

there. We say that the pair (X, D) is **regular**, or that (X, D) is a **regular pair**, if (X, D) is regular at all $P \in X$.

Definition 2.7. Let (X,D) be a regular pair, and let $f: X \to Y$ be a dominant morphism. We say that (X,D) has **good reduction** at a point $Q \in Y$ if the pair $(X_Q,D|_{X_Q})$ is regular, where X_Q is the fiber of f over Q.

Proposition 2.8. Let $f: X \to Y$ be a dominant morphism of integral, separated schemes of finite type over \mathbb{Z} , and let D be a divisor on X. If the pair (X, D) is regular, then it has good reduction over an open dense subset of Y.

Proof. First note that (X, D) has good reduction over the generic point of Y. Indeed, the condition of Definition 2.4 remains true when restricting to the generic fiber. Thus it will suffice to show that the set

$$\left\{P \in X : (X_{f(P)}, D\big|_{X_{f(P)}}) \text{ is regular at } P\right\}$$

is an open subset of X. But this follows from the corresponding fact for smooth morphisms, via Remark 2.5.

Lemma 2.9. Let $i: G \hookrightarrow \overline{G}$ be an equivariant completion of a semiabelian variety over an algebraically closed field of characteristic zero, such that the pair $(\overline{G}, \overline{G} \setminus G)$ is regular. Then the logarithmic canonical line sheaf $\mathscr{K}_{\overline{G}}(\overline{G} \setminus G)$ is trivial.

Proof. This was proved in the first few sentences of the proof of ([Vo 3], Lemma 5.6), but under the additional assumption that the equivariant completion is of the form indicated in ([Vo 3], Lemma 2.2). It remains to be shown that all equivariant completions of semiabelian varieties are of this form (*i.e.*, obtained in a natural way from an equivariant completion of the toric part).

Let $\rho\colon G\to A$ be the maximal abelian quotient of G, and let k be the ground field. Since \overline{G} is a nonsingular variety, ρ extends to a morphism $\overline{\rho}\colon \overline{G}\to A$ ([Mi], Thm. 3.1). Since k is algebraically closed, the kernel of ρ is a split torus, so every point of A has a Zariski-open neighborhood U such that $\rho^{-1}(U)$ is a product $U\times \mathbb{G}_{\mathrm{m}}^{\mu}$. In particular, ρ has a rational section over U. By equivariance, $\overline{\rho}^{-1}(U)$ must therefore be of the form $\overline{\rho}^{-1}(0)\times U$, hence \overline{G} is of the desired form.

Proposition 2.10. Let k, S, X, D, \mathscr{A} , and ϵ be as in Conjecture A. Assume:

- (i). a semiabelian variety G of dimension $\dim X 1$ acts faithfully on X, preserving D;
- (ii). the support of D contains all degenerate orbits;
- (iii). the map $\pi\colon U\to V$ of Lemma 2.3 extends to a morphism $\bar\pi\colon X\to \overline V$ for some projective completion $\overline V$ of V; and
- (iv). Conjecture A holds for the above data.

Then Conjecture A holds for the above data with $Z = \bar{\pi}^{-1}(\Sigma)$, where Σ is the set of all points on \overline{V} over which the pair (X, D) has bad reduction.

Proof. Let Z be an exceptional set for which Conjecture A holds.

There exists a finite subset $\Gamma \subseteq G(k)$ such that

$$Z' := \bigcap_{g \in \Gamma} \left(T_g^{-1}(Z) \cup T_{-g}^{-1}(Z) \right)$$

is preserved under the action of G, where $T_g\colon X\to X$ denotes translation by g. This can be rewritten

$$Z' = \bigcup_{\sigma} \bigcap_{g \in \Gamma} T_{\sigma(g)g}^{-1}(Z) ,$$

where σ ranges over all functions $\sigma \colon \Gamma \to \{\pm 1\}$.

Since the height and proximity functions are functorial, condition (iv) implies that for any fixed $g \in G$,

(2.10.1)
$$m_{S,k}(T_q^*D, P) + h_{\mathcal{K},k}(P) \le \epsilon h_{T_q^*\mathcal{A},k}(P) + O(1)$$

for all $P \in (X \setminus T_g^*(Z))(k)$ (with a possibly different constant O(1), depending on g). Clearly $T_g^*D = D$. Moreover,

$$T_q^* \mathscr{A} \otimes T_{-q}^* \mathscr{A} \cong \mathscr{A}^{\otimes 2}$$
.

Indeed, this holds because $g \mapsto T_g^* \mathscr{A} \otimes \mathscr{A}^{\vee}$ defines a morphism $G \to \operatorname{Pic}^0(X)$, which must be a group homomorphism. Thus,

$$h_{\mathscr{A},k}(P) \geq \min\{h_{T_g^*\mathscr{A},k}(P),h_{T_{-g}^*\mathscr{A},k}(P)\} + O(1)$$

for all $P \in X(\bar{k})$, where the constant O(1) depends on g but not on P. Consequently, we have

$$h_{\mathscr{A},k}(P) \ge \max_{g \in \Gamma} \min\{h_{T_g^*\mathscr{A},k}(P), h_{T_{-g}^*\mathscr{A},k}(P)\} + O(1)$$

$$\ge \min_{\sigma \in \{\pm 1\}^{\Gamma}} \max_{g \in \Gamma} h_{T_{\sigma(g)g}^*\mathscr{A},k}(P) .$$

Thus (2.10.1), applied for all g such that $\pm g \in \Gamma$, implies that Conjecture A holds with exceptional set Z' (after adjusting O(1)). Therefore we may assume that Z is preserved by G.

Let Y be an irreducible component of Z not contained in $\operatorname{Supp} D$. Then Y is preserved by the group action, so it contains a non-degenerate orbit. For dimension reasons, it follows that Y is the closure of a non-degenerate orbit. Thus Y is an equivariant completion of a semiabelian variety, and by the assumptions on D, it follows that the semiabelian variety of which Y is the closure, is the complement of $Y \cap \operatorname{Supp} D$.

In addition, Y is contained in a fiber of $\bar{\pi}$. Indeed, let G act trivially on V and on \overline{V} . Then π is G-equivariant, so the same is true for $\bar{\pi}$. Thus fibers of $\bar{\pi}$ are G-invariant.

Suppose now that Y does not lie on a fiber of bad reduction for (X,D). Then, for dimension reasons (since \overline{V} is a curve) Y is the entire fiber of $\overline{\pi}$, and the pair $(Y,D|_Y)$ is regular since the fiber has good reduction. As mentioned above, Y is an equivariant completion of a semiabelian variety, so Lemma 2.9 implies that the line sheaf $\mathscr{K}_Y(D|_Y)$ is trivial. It then follows easily that Conjecture A holds on Y with empty exceptional set. But also, since Y is an entire fiber, we have that $\mathscr{O}(Y)|_Y$ is trivial, so $\mathscr{K}_Y \cong \mathscr{K}_X|_Y$, and therefore Conjecture A on Y implies that all points on Y satisfy Conjecture A on X (after suitably adjusting the $\mathscr{O}(1)$ term). Thus Y can be removed from Z.

Remark 2.11. It is not always possible to take $Z=\emptyset$ in Conjecture B, as the following example illustrates. Let X be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the point (0,1), and let D be the pull-back of the divisor $[0] \times \mathbb{P}^1 + [\infty] \times \mathbb{P}^1 + \mathbb{P}^1 \times [0] + \mathbb{P}^1 \times [\infty]$. Let \mathbb{G}_{m} act on $\mathbb{P}^1 \times \mathbb{P}^1$ by acting in the obvious way on the first factor; this extends to an action of \mathbb{G}_{m} on X that preserves D. In this situation, Conjecture A does not holds for (X,D) unless Z contains the strict transform Y of $\mathbb{P}^1 \times \{1\}$. This is true because the restriction $\mathscr{K}_X|_Y$ is isomorphic to $\mathscr{O}(-1)$ instead of $\mathscr{O}(-2)$ (via the isomorphism $Y \cong \mathbb{P}^1$).

Remark 2.12. Conditions (ii) and (iii) do not necessarily restrict the applicability of Proposition 2.10. Given a regular pair (X, D) with a group action as in (i), one can find another regular pair (X', D') satisfying (i)–(iii), admitting a proper birational morphism $\pi \colon X' \to X$ such that $\operatorname{Supp} D' \supseteq \operatorname{Supp} \pi^*D$ and such that the group action extends to X' and preserves D'. In that situation, Conjecture A for (X', D') implies Conjecture A for (X, D), so if Conjecture A is known for (X', D'), then the exceptional set for (X, D) will be contained in the image of the exceptional set for (X', D').

§3. Conjecture B and the abc conjecture

This section shows that Conjecture B, if proved, would imply the abc conjecture.

In this section, all heights, proximity functions, and counting functions are taken relative to $S = \{\infty\}$ over \mathbb{Q} , so the subscripts will be omitted from the notation. The abc conjecture can also be formulated over number fields, with arbitrary (finite) set S. The methods of this sections extend readily to this case.

Definition 3.1. For $\ell \in \mathbb{Z}$, $\ell \neq 0$, let

$$n(\ell) = \sum_{p|\ell} \log p$$

(where the sum is over distinct primes dividing ℓ).

Then the main inequality of the abc conjecture can be replaced by

$$(3.2) n(abc) \ge (1 - \epsilon)h([a:b:c]) - O_{\epsilon}(1)$$

(with a different ϵ). We will use this inequality in place of (0.1.1).

The basic idea of this section is to construct, for a given ϵ , a regular pair (Γ', D') , and for each triple (a, b, c) as in the abc conjecture a point $P'_{a,b,c} \in \Gamma'(\mathbb{Q})$ such that Conjecture B applied to the points $P'_{a,b,c}$ implies the abc conjecture with the desired value of ϵ .

To begin, fix $0 < \epsilon < 1$, and let n be an integer with $n > 3/\epsilon$.

For triples $a,b,c\in\mathbb{Z}$ with a+b+c=0 and (a,b,c)=1, let x_1,\ldots,x_n be integers such that

$$x_1 x_2^2 x_3^3 \cdots x_n^n = a$$

and such that

$$\operatorname{ord}_{p}(x_{i}) = \begin{cases} \left[\frac{\operatorname{ord}_{p}(a)}{n}\right] & \text{if } i = n;\\ 1 & \text{if } i = \operatorname{ord}_{p}(a) - n\left[\frac{\operatorname{ord}_{p}(a)}{n}\right]; \text{ and}\\ 0 & \text{otherwise.} \end{cases}$$

Likewise, choose y_1, \ldots, y_n and z_1, \ldots, z_n such that

$$y_1 y_2^2 y_3^3 \cdots y_n^n = b$$
 and $z_1 z_2^2 z_3^3 \cdots z_n^n = c$.

This defines a point $P = P_{a,b,c}$ in $(\mathbb{P}^2)^n$ with multihomogeneous coordinates

$$([x_1:y_1:z_1],[x_2:y_2:z_2],\ldots,[x_n:y_n:z_n])$$
.

All such points lie on the variety $X_n \subseteq (\mathbb{P}^2)^n$ defined by the equation

$$\prod_{i=1}^{n} x_i^i + \prod_{i=1}^{n} y_i^i + \prod_{i=1}^{n} z_i^i = 0.$$

We will want to compare the height of $P_{a,b,c} \in X_n$ with the height h([a:b:c]) occurring in (3.2). This can be done via functoriality of heights, as follows. The map

$$([x_1:y_1:z_1],\ldots,[x_n:y_n:z_n]) \mapsto \left(\prod x_i^i:\prod y_i^i:\prod z_i^i\right)$$

defines a rational map $X_n \dashrightarrow \mathbb{P}^2$ (actually to the line a+b+c=0 in \mathbb{P}^2). Let Γ_n be the closure of the graph of this rational map, and let $\phi \colon \Gamma_n \to \mathbb{P}^2$ be the corresponding morphism. Then $\Gamma_n \subseteq \left(\mathbb{P}^2\right)^{n+1}$ is defined by the equations:

$$(3.3a) \qquad \qquad \prod x_i^i + \prod y_i^i + \prod z_i^i = 0$$

$$(3.3b) a+b+c=0$$

$$(3.3c) a \prod y_i^i = b \prod x_i^i.$$

This system appears to be asymmetrical in the variables, but in fact we have

$$0 = a \left(\prod x_i^i + \prod y_i^i + \prod z_i^i \right) - (a+b+c) \prod x_i^i$$
$$= \left(a \prod y_i^i - b \prod x_i^i \right) + \left(a \prod z_i^i - c \prod x_i^i \right)$$

and similarly for $b \prod z_i^i - c \prod y_i^i$. Thus Γ_n is preserved by the symmetric group \mathscr{S}_3 acting by permuting the variables.

Given a triple (a, b, c) as above, let $P^* = P^*_{a,b,c}$ be the (unique) point on Γ_n lying over $P_{a,b,c} \in X_n$. Then we have

(3.4)
$$h([a:b:c]) = h_{\phi^*\mathscr{O}(1)}(P^*) + O(1) .$$

From now on, let D be the divisor $x_1 \cdots x_n y_1 \cdots y_n z_1 \cdots z_n = 0$ on Γ_n , and let E be the divisor [1:-1:0] + [0:1:-1] + [-1:0:1] on the line a+b+c=0 in \mathbb{P}^2 . The next step is to compare n(abc) with $N(D, P_{a,b,c}^*)$.

Lemma 3.5. For relatively prime integers a, b, c with a + b + c = 0,

(3.5.1)
$$N(D, P_{a,b,c}^*) \le n(abc) + \frac{1}{n} N(E, \phi(P_{a,b,c}^*)) + O(1) .$$

Proof. To begin, note that

$$\lambda_{E,v}([a:b:c]) := -\log \frac{\|abc\|_v}{\max\{\|a\|_v, \|b\|_v, \|c\|_v\}^3}$$

and

$$\lambda_{D,v}([a:b:c], [x_1:y_1:z_1], \dots, [x_n:y_n:z_n])$$

$$:= -\log \frac{\|x_1 \cdots x_n y_1 \cdots y_n z_1 \cdots z_n\|_v}{\max\{\|x_1\|_v, \|y_1\|_v, \|z_1\|_v\}^3 \cdots \max\{\|x_n\|_v, \|y_n\|_v, \|z_n\|_v\}^3}$$

are Weil functions for E on a+b+c=0 in \mathbb{P}^2 and for D on X, respectively. We assume here that a,b,c are relatively prime and that x_i,y_i,z_i are relatively prime for all i. Under that assumption, the above expressions simplify to

$$\lambda_{E,v}([a:b:c]) = \operatorname{ord}_{v}(abc) \log v$$

and

$$\lambda_{D,v}([a:b:c],[x_1:y_1:z_1],\ldots,[x_n:y_n:z_n]) = \operatorname{ord}_p(x_1\cdots x_ny_1\cdots y_nz_1\cdots z_n)\log p$$

respectively, for places v corresponding to rational primes p. We use these Weil functions to define $N(D, P^*_{a,b,c})$ and $N(E, \phi(P^*_{a,b,c}))$.

With these choices, we show that (3.5.1) holds without the O(1) term. Each term is a sum over all rational primes p, so it will suffice to show that the inequality holds for each p. Fix a prime p. If $p \nmid abc$, then p does not contribute anything to any of these terms, so the inequality holds for p. Otherwise, by symmetry we may assume that $p \mid a$; hence $p \nmid b$ and $p \nmid c$. The contribution to the left-hand side at p is then

$$\begin{cases} \frac{\operatorname{ord}_p a}{n} \log p & \text{if } n \mid \operatorname{ord}_p a; \\ \left(\left[\frac{\operatorname{ord}_p a}{n} \right] + 1 \right) \log p & \text{if } n \nmid \operatorname{ord}_p a. \end{cases}$$

The contribution on the right-hand side is $\left(1 + \frac{\operatorname{ord}_p a}{n}\right) \log p$; hence the inequality holds for the terms at p in this case, too.

Corollary 3.6.
$$N(D, P_{a,b,c}^*) \le n(abc) + \frac{3}{n}h(\phi(P_{a,b,c}^*)) + O(1)$$
.

Proof. This is immediate from the definitions of the height and counting functions on \mathbb{P}^2 .

At this point we note that the group $G := \mathbb{G}_{\mathrm{m}}^{2n-2}$ acts faithfully on the pair (Γ_n, D) , via

$$(3.8) (u_2, \dots, u_n, v_2, \dots, v_n) \cdot ([a:b:c], [x_1:y_1:z_1], \dots, [x_n:y_n:z_n])$$

$$= ([a:b:c], [x_1u_2^{-2}u_3^{-3} \cdots u_n^{-n}: y_1v_2^{-2} \cdots v_n^{-n}: z_1],$$

$$[x_2u_2: y_2v_2: z_2], \dots, [x_nu_n: y_nv_n: z_n]).$$

One would like to apply Conjecture B to the pair (Γ_n, D) , but it is not regular. However, we do have the following.

Lemma 3.7. The pair (Γ_n, D) is regular outside of the set

$$(3.7.1) x_1 \cdots x_n = y_1 \cdots y_n = z_1 \cdots z_n = 0.$$

Proof. Let P be a point on Γ_n with multihomogeneous coordinates

$$([a:b:c],[x_1:y_1:z_1],\ldots,[x_n:y_n:z_n])$$
.

It is a regular point of Γ_n if and only if the matrix

$$\begin{bmatrix} 0 & 0 & 0 & \frac{\prod x_i^i}{x_1} & \dots & \frac{n \prod x_i^i}{x_n} & \frac{\prod y_i^i}{y_1} & \dots & \frac{n \prod y_i^i}{y_n} & \frac{\prod z_i^i}{z_1} & \dots & \frac{n \prod z_i^i}{z_n} \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \prod y_i^i & -\prod x_i^i & 0 & \frac{-b \prod x_i^i}{x_1} & \dots & \frac{-nb \prod x_i^i}{x_n} & \frac{a \prod y_i^i}{y_1} & \dots & \frac{na \prod y_i^i}{y_n} & 0 & \dots & 0 \end{bmatrix}$$

has rank 3. Assume that (3.7.1) does not hold at P. By symmetry, we may assume that $x_1 \cdots x_n \neq 0$. Then the second, third, and fourth columns are linearly independent, so Γ_n is regular at P.

It remains to be shown that D has normal crossings at such points P. This only needs to be checked if $P \in \operatorname{Supp} D$. Again assume that $x_1 \cdots x_n \neq 0$; since $P \in \operatorname{Supp} D$ we may assume by symmetry that $z_1 \cdots z_n = 0$. Then, by (3.3a), $y_1 \cdots y_n \neq 0$. Then the rational map $\Gamma_n \dashrightarrow \mathbb{A}^{2n-1}$ defined by

$$([a:b:c],[x_1:y_1:z_1],\ldots,[x_n:y_n:z_n]) \mapsto \left(\frac{x_2}{y_2},\ldots,\frac{x_n}{y_n},\frac{z_1}{y_1},\ldots,\frac{z_n}{y_n}\right)$$

is regular at P. Then the first, third, and fourth columns of the matrix mentioned earlier are linearly independent, implying that the above map is étale at P. Thus D has normal crossings at P.

The following lemma constructs a well-behaved resolution of the above singularities.

Lemma 3.9. There exist a complete variety Γ'_n , a divisor D' on Γ'_n , and a birational morphism $\psi \colon \Gamma'_n \to \Gamma_n$ such that:

- (i). Supp $D' = \psi^{-1}(\operatorname{Supp} D)$,
- (ii). the pair (Γ'_n, D') is regular,
- (iii). ψ is an isomorphism over the set

(3.9.1)
$$\left\{P \in \Gamma_n : (\Gamma_n, D) \text{ is regular at } P \text{ and } \phi(P) \notin \operatorname{Supp} E\right\},$$

(iv). $\psi^*\phi^*\mathcal{O}(1) \leq \mathcal{K}_{\Gamma'_n} + D'$ relative to the cone of effective divisors on Γ'_n , and (v). the group action (3.8) extends to an action on (Γ'_n, D') .

Proof. We start by noting that if [a:b:c] and [a':b':c'] are points on \mathbb{P}^2 with a+b+c=a'+b'+c'=0 and $[a:b:c], [a',b',c'] \notin \operatorname{Supp} E$, then the morphism

$$([a:b:c],[x_1:y_1:z_1],\ldots,[x_n:y_n:z_n])$$

$$\mapsto ([a':b':c'],[a'bcx_1:ab'cy_1:abc'z_1],[x_2:y_2:z_2],\ldots,[x_n:y_n:z_n])$$

induces an isomorphism $\phi^{-1}([a:b:c]) \cong \phi^{-1}([a':b':c'])$. Moreover, this isomorphism preserves the restriction of D to the fibers, as well as the group action.

Let F be one such fiber, and let D_F denote the restriction of D to F. Also, identify \mathbb{P}^1 with the line a+b+c=0 in \mathbb{P}^2 . Then the above isomorphisms define a rational map $\Gamma_n \to F \times \mathbb{P}^1$. This rational map is an isomorphism away from the set $\phi^{-1}(\operatorname{Supp} E)$. Moreover, it is G-equivariant.

Let $\rho \colon F^* \to F$ be an embedded resolution of D_F on F. By [**B-M**] we may choose ρ such that the action of G on F extends to F^* and such that ρ is an isomorphism over the set where (F, D_F) is regular. This defines a rational map $\Gamma_n \dashrightarrow F^* \times \mathbb{P}^1$. Let Γ_n^* be the closure of the graph of this map, with projections $\alpha \colon \Gamma_n^* \to \Gamma_n$ and $\tau \colon \Gamma_n^* \to F^* \times \mathbb{P}^1$. Then α is an isomorphism over the set (3.9.1), and τ is an isomorphism away from $(\phi \circ \alpha)^{-1}(\operatorname{Supp} E)$.

Finally, let $\beta \colon \Gamma_n' \to \Gamma_n^*$ be an embedded resolution of $\alpha^{-1}(\operatorname{Supp} D)$ in Γ_n^* , let D' be the resulting normal crossings divisor on Γ_n' , and let $\psi = \alpha \circ \beta$. Again, we may choose β so that the group action extends and such that β is an isomorphism over the open set where $(\Gamma_n^*, \alpha^{-1}(\operatorname{Supp} D))$ is regular.. Thus the pair (Γ_n', D') satisfies conditions (i), (ii), and (v). Condition (iii) also holds, since the same is true for α and β .

$$\begin{array}{ccc}
\Gamma'_{n} \\
\beta \downarrow \\
\Gamma^{*}_{n} & \xrightarrow{\tau} & F^{*} \times \mathbb{P}^{1} \\
\alpha \downarrow & & \downarrow \rho \times \mathrm{Id}_{\mathbb{P}^{1}} \\
\Gamma_{n} & \xrightarrow{-\to} & F \times \mathbb{P}^{1}
\end{array}$$

Finally, consider Condition (iv). To begin, note that if a=0 at some point $P\in\Gamma_n$, then (3.3a) implies that $\prod x_i=0$, so $P\in\operatorname{Supp} D$. Let D_{F^*} be the normal crossings divisor lying over D_F . Then

$$\operatorname{Supp} D' = (\tau \circ \beta)^{-1} \big((\operatorname{Supp} D_{F^*} \times \mathbb{P}^1) \cup (F^* \times \operatorname{Supp} E) \big) .$$

Since F^* is a toric variety with principal orbit $F^* \setminus D_{F^*}$, Lemma 2.9 implies that its logarithmic canonical line sheaf $\mathscr{K}_{F^*}(D_{F^*})$ is trivial; hence the logarithmic canonical divisor of $(F^* \times \mathbb{P}^1) \setminus (D_{F^*} \times \mathbb{P}^1 + F^* \times E)$ is the pull-back of $\mathscr{O}(1)$ from the second factor. Thus the line sheaf $\mathscr{K}_{\Gamma'_n}(D') \otimes \psi^* \phi^* \mathscr{O}(-1)$ is the line sheaf associated to the logarithmic ramification divisor of $\tau \circ \beta$. This divisor is effective, so Condition (iv) holds.

By construction and Lemma 3.7, ψ is an isomorphism over the point $P_{a,b,c}^* \in \Gamma_n$ for all triples a,b,c of relatively prime integers with a+b+c=0, so there is a well-defined point $P'_{a,b,c} = \psi^{-1}(P^*_{a,b,c})$ in Γ'_n .

Lemma 3.10. For relatively prime integers a, b, c with a + b + c = 0,

$$N(D', P'_{a,b,c}) \le N(D, P^*_{a,b,c}) + O(1)$$
.

Proof. Since D' is reduced and $\operatorname{Supp} D' = \psi^{-1}(\operatorname{Supp} D)$, the divisor $\psi^*D - D'$ is effective. The inequality then follows from additivity and functoriality properties of the counting function.

Lemma 3.11. For relatively prime integers a, b, c with a + b + c = 0,

$$h_{\mathscr{O}(1,1,...,1)}(P_{a,b,c}^*) \leq 4h([a:b:c]) + O(1) \; .$$

Proof. By construction

$$P_{a,b,c}^* = ([a:b:c], [x_1:y_1:z_1], \dots, [x_n:y_n:z_n]),$$

where the x_i , y_i , and z_i are as in the construction of $P_{a,b,c}$. In particular, x_i, y_i, z_i are triples of relatively prime integers for each i. Thus, we may take

$$h_{\mathcal{O}(1,1,\ldots,1)}(P_{a,b,c}^*) = \log \max\{|a|,|b|,|c|\} + \sum_{i=1}^n \log \max\{|x_i|,|y_i|,|z_i|\}.$$

The first term equals h([a:b:c]), and it is easy to see from the construction that the second term is bounded from above by 3h([a:b:c]).

We are now ready to state and prove the main theorem of the section.

Theorem 3.12. If, for some n, Conjecture B holds for the pair (Γ'_n, D') over \mathbb{Q} , then the abc conjecture holds for all $\epsilon > 3/n$.

Proof. Let $\mathscr{A} = \psi^* \mathscr{O}(1,1,\ldots,1)$. Since it is the pull-back of a big line sheaf by a birational morphism, it is big. Pick ϵ' such that $4\epsilon' \leq \epsilon - 3/n$. Then, by (3.4), Condition (iv) of Lemma 3.9, properties of heights and proximity and counting functions, the assumption that Conjecture B holds, Lemma 3.10, Corollary 3.6, and Lemma 3.11, we have:

$$\begin{split} h([a:b:c]) &= h_{\psi^*\phi^*\mathscr{O}(1)}(P'_{a,b,c}) + O(1) \\ &\leq h_{\mathscr{K}_{\Gamma'_n}(D')}(P'_{a,b,c}) + O(1) \\ &= m(D',P'_{a,b,c}) + h_{\mathscr{K}_{\Gamma'_n}}(P'_{a,b,c}) + N(D',P'_{a,b,c}) + O(1) \\ &\leq \epsilon' \, h_{\mathscr{A}}(P'_{a,b,c}) + N(D,P^*_{a,b,c}) + O(1) \\ &\leq \epsilon' \, h_{\mathscr{A}}(P'_{a,b,c}) + n(abc) + \frac{3}{n} h(P^*_{a,b,c}) + O(1) \\ &\leq n(abc) + \epsilon \, h([a:b:c]) + O(1) \end{split}$$

for all a,b,c such that $P'_{a,b,c}$ lies outside of a certain proper Zariski-closed subset Z of Γ'_n . This set is the union of the exceptional set for Conjecture B and the base locus of the effective divisor implicit in Lemma 3.9(iv). As in Section 2, we may reduce to the case where Z is G-invariant, so (after ignoring components contained in $\operatorname{Supp} D'$) it is a finite union of fibers of $\phi \circ \psi \colon \Gamma'_n \to \mathbb{P}^1$. Each fiber can contain at most two points $P'_{a,b,c}$ (corresponding to a,b,c and -a,-b,-c), so the exceptional set can be eliminated by adjusting the O(1) term.

Corollary 3.13. If Conjecture B holds for all pairs (X, D) over \mathbb{Q} , then the abc conjecture holds.

§4. Conjecture B in the Split Function Field Case

This section shows that, in the split function field case of characteristic zero, the counterpart to Conjecture B holds. We begin with a definition.

Definition 4.1. Let C be a smooth projective curve over an algebraically closed field, let S be a subset of C, and let $D = \sum_{P \in C} n_P \cdot [P]$ be a divisor on C. Then

$$\deg_S D = \sum_{P \in S} n_P \ .$$

Theorem 4.2. Let k_0 be an algebraically closed field, and let (X, D) be a regular pair with X complete over k_0 . Assume that a semiabelian variety G of dimension $\dim X - 1$ acts faithfully on (X, D). Let C be a smooth projective curve over k_0 and let S be a finite set of points on C. Then there exists a proper Zariski-closed subset Z of X such that any map $f \colon C \to X$ with $\operatorname{Im} f \nsubseteq Z \cup \operatorname{Supp} D$ satisfies the inequality

(4.2.1)
$$\deg_{C \setminus S}(f^*D)_{\text{red}} \ge \deg f^* \mathscr{K}_X(D) - \max\{0, 2g(C) - 2 + \#S\}$$

and therefore the inequality

$$(4.2.2) \deg_S f^* \mathcal{X}_X + \deg_S f^* D \le \max\{0, 2g(C) - 2 + \#S\}.$$

The left-hand side of the first inequality is the counterpart, in the split function field case, to the truncated counting function $N^{(1)}(D,\cdot)$. Conjecture A has also been posed for truncated counting functions [Vo 2].

Proof. As was noted in Section 2, we may assume that the map $\pi\colon U\to V$ of Lemma 2.3 extends to a morphism $\bar\pi\colon X\to \overline V$ for some projective completion $\overline V$ of V, and that Supp D contains all degenerate orbits. In this case these changes do not weaken the inequality at all, even up to O(1). Let Z be as in Proposition 2.10.

It will suffice to prove (4.2.1), since (4.2.2) follows immediately by applying the trivial inequality

$$\deg_{C \setminus S} (f^*D)_{\text{red}} \le \deg_{C \setminus S} f^*D = \deg f^*D - \deg_S f^*D .$$

Consider the diagram

in which α is the map $(\gamma, x) \mapsto (\gamma x, x)$. This induces a map of relative tangent bundles:

$$T_{G \times X/X} \xrightarrow{\alpha_*} \alpha^* T_{X \times X/X} \cong \alpha^* \operatorname{pr}_1^* T_X \cup$$

$$\alpha^* \operatorname{pr}_1^* (T_X(-\log D)).$$

This is a map of sheaves on $G \times X$. Since the action of G preserves D, the image actually lies in $\alpha^* \operatorname{pr}_1^* (T_X(-\log D))$. Let $n = \dim X$. Taking \wedge^{n-1} of everything gives:

$$\mathscr{O}_{G\times X} \cong \wedge^{n-1} T_{G\times X}/X \xrightarrow{\alpha_*} \alpha^* \operatorname{pr}_1^* \wedge^{n-1} T_X(-\log D)$$
.

Pulling back via the map $(0, \mathrm{Id}_X): X \to G \times X$ gives:

$$\mathcal{O}_X \to (0, \operatorname{Id}_X)^* \alpha^* \operatorname{pr}_1^* \wedge^{n-1} T_X(-\log D)$$
$$= \wedge^{n-1} T_X(-\log D)$$
$$\cong \Omega_X^1(\log D) \otimes (\mathcal{X}_X(D))^{\vee}.$$

Since G acts faithfully, the above map is nonzero, so it determines a nonzero global section

$$\omega \in \Gamma(X, \Omega_X^1(\log D) \otimes (\mathscr{K}_X(D))^{\vee})$$
.

The above construction is essentially the same as in Section 2, except that the divisor D has been added.

Now consider $f^*\omega$.

Case I. If $f^*\omega=0$ then f is everywhere tangent to an orbit under the G-action; hence $\operatorname{Im} f$ is contained in the closure of some fixed orbit Y. By assumption Y is a non-degenerate orbit. By definition of Z, Y lies on a fiber of good reduction for $\bar{\pi}$, so the closure \overline{Y} is the whole fiber, the pair $(\overline{Y},D|_{\overline{Y}})$ is regular, and the adjunction formula gives $\mathscr{K}_{\overline{Y}}\cong\mathscr{K}_X|_{\overline{Y}}$. But \overline{Y} is an equivariant completion of a semiabelian variety $\overline{Y}\setminus D|_{\overline{Y}}$, so by Lemma 2.9,

$$\mathscr{K}_X(D)\big|_{\overline{Y}} \cong \mathscr{K}_{\overline{Y}}\big(D\big|_{\overline{Y}}\big) \cong \mathscr{O}_{\overline{Y}};$$

hence $f^*\mathcal{K}_X(D)$ is trivial. This gives (4.2.1) since, on the left-hand side, f^*D is effective.

Case II. Assume $f^*\omega \neq 0$. Then it determines a nonzero global section of

$$\mathscr{K}_C(\log f^*D)\otimes f^*(\mathscr{K}_X(D))^{\vee}$$
.

Taking degrees, this implies that

$$2g(C) - 2 + \deg(f^*D)_{\text{red}} - \deg f^* \mathcal{K}_X(D) \ge 0.$$

Applying the inequality

$$\#S \ge \deg_S(f^*D)_{\mathrm{red}}$$

then gives (4.2.1).

§5. Conjecture B for holomorphic curves

This section shows that Conjecture B holds in the case of holomorphic curves (*i.e.*, Nevanlinna theory). This is done by methods analogous to those of Section 4. This relies on a result of McQuillan to replace the simple argument based on comparing degrees of line sheaves.

Proposition 5.1. Let (X,D) be a regular pair with X complete over \mathbb{C} , let $f:\mathbb{C} \to X$ be a holomorphic curve not lying entirely in $\operatorname{Supp} D$, let $f':\mathbb{C} \to \mathbb{P}(\Omega^1_X(\log D))$ be its derivative, let $\mathscr{O}(1)$ be the tautological line sheaf on $\mathbb{P}(\Omega^1_X(\log D))$, and let \mathscr{A} be a line sheaf on X whose restriction to the Zariski closure of the image of f is big. Then

$$T_{\mathscr{O}(1),f'}(r) \leq_{\mathrm{exc}} N_f^{(1)}(D,r) + O(\log^+ T_{\mathscr{A},f}(r)) + o(\log r)$$
.

Here the notation \leq_{exc} means that the inequality holds outside of a set of finite Lebesgue measure, and $N_f^{(1)}(D,r)$ denotes the truncated counting function.

When D=0, this was proved by McQuillan ([McQ 2], Theorem A). A proof of the general case is to appear in [McQ 3]. Appendix A of this paper gives a more classical proof, based on the lemma on the logarithmic derivative, using ideas from P.-M. Wong [W].

Corollary 5.2. Let X, D, f, f', and $\mathscr A$ be as above, let $\mathscr L$ be a line sheaf on X, let $d \in \mathbb Z_{>0}$, and let ω be a global section of $S^d\big(\Omega^1_X(\log D)\big) \otimes \mathscr L^\vee$. If $f^*\omega \neq 0$, then

$$T_{\mathcal{L},f}(r) \leq_{\text{exc}} d \cdot N_f^{(1)}(D,r) + O(\log^+ T_{\mathcal{A},f}(r)) + o(\log r)$$
.

Proof. Let $p: \mathbb{P}(\Omega_X^1(\log D)) \to X$ be the canonical projection. The section ω corresponds to a global section

$$\omega' \in \Gamma(\mathbb{P}(\Omega_X^1(\log D)), \mathcal{O}(d) \otimes p^* \mathcal{L}^{\vee})$$
,

and $f'^*\omega' = f^*\omega$. Thus the image of f' is not contained in the base locus of $\mathcal{O}(d) \otimes p^*\mathcal{L}^\vee$, so

$$T_{\mathscr{O}(d),f'}(r) \ge T_{\mathscr{L},f}(r) + O(1)$$
.

The above corollary now makes it easy to prove the following counterpart to Conjecture B in Nevanlinna theory.

Theorem 5.3. Let (X,D) be a regular pair with X complete over \mathbb{C} . Assume that a semiabelian variety G of dimension $\dim X-1$ acts faithfully on (X,D). Let \mathscr{A} be a big line sheaf on X. Then there is a proper Zariski-closed subset $Z\subseteq X$ such that any holomorphic curve $f\colon \mathbb{C}\to X$ with $\mathrm{Im}\, f\nsubseteq Z\cup \mathrm{Supp}\, D$ satisfies the inequality

(5.3.1)
$$N_f^{(1)}(D,r) \ge_{\text{exc}} T_{\mathcal{K}_X(D),f}(r) - O(\log^+ T_{\mathcal{A},f}(r)) - o(\log r)$$

and therefore also the inequality

$$(5.3.2) T_{\mathcal{X}_X,f}(r) + m_f(D,r) \leq_{\text{exc}} O(\log^+ T_{\mathcal{A},f}(r)) + o(\log r)$$

Proof. As in Section 4, we may assume that the map $\pi\colon U\to V$ of Lemma 2.3 extends to a morphism $\bar\pi\colon X\to \overline V$ for some projective completion $\overline V$ of V, and that $\operatorname{Supp} D$ contains all degenerate orbits. These changes weaken the inequalities only up to O(1). Let Z_0 be as in Z of Proposition 2.10.

Let $\pi\colon X'\to X$ be a birational morphism, with X' projective. By Kodaira's lemma (see, for example, ([Vo 1], Prop. 1.2.7)) some positive tensor power of $\pi^*\mathscr{A}$ is isomorphic to $\mathscr{A}'\otimes\mathscr{O}(D)$, with \mathscr{A}' ample and D effective. Let Z_1 be the image under π of the base locus of D. Then let $Z=Z_0\cup Z_1$, and assume that the image of f is not contained in $Z\cup\operatorname{Supp} D$.

Also as in Section 4, it will suffice to prove (5.3.1).

Let ω be the form constructed in the proof of Theorem 4.2. As in that proof, if $f^*\omega = 0$, then the image of ω is contained in the closure of a non-degenerate orbit Y of the group action, and (5.3.1) holds again since $f^*\mathcal{K}_X(D)$ is trivial.

Otherwise, $f^*\omega \neq 0$, and (5.3.1) follows immediately from Corollary 5.2.

§6. A hierarchy of problem types

For several decades, it has been known that valuable insight into a diophantine problem over number fields can be gained by looking at the corresponding problem over function fields. In the function field case, one looks first at the split case (where everything is defined over the field of constants of the function field).

More recently, it has been observed that diophantine problems are formally similar to problems in Nevanlinna theory, and that insight into the former may be gained by looking at the latter (and sometimes vice versa).

This section introduces a hierarchy of problem types that incorporates the above observations, plus a few others.

Given a classification of regular pairs (X, D), one can pose a number of related problems in various contexts:

- Find the (algebraic) **exceptional set**; i.e., the Zariski closure of the union of the images of all non-constant strictly rational maps $G \dashrightarrow X \setminus D$, where G is either \mathbb{G}_{m} or an abelian variety. A **strictly rational map** ([I], §2.12) is a rational map $X \dashrightarrow Y$ such that the closure of the graph is proper over X.
- For each $\epsilon > 0$, find the exceptional subset Z for the main inequality of Conjecture A.
- Prove the inequality of Conjecture A in the split function field case of characteristic zero.
- Prove that all non-constant holomorphic curves $\mathbb{C} \to X \setminus D$ must lie in the exceptional set.
- Prove the inequality of Conjecture A for holomorphic curves $\mathbb{C} \to X$.

- Prove that the set of integral points on $X \setminus D$ is not Zariski-dense, in the (general) function field case of characteristic zero.
- Prove the inequality of Conjecture A in the function field case of characteristic zero.
- Prove that the set of integral points on $X \setminus D$ is not Zariski-dense, in the function field case of characteristic p > 0.
- Prove the inequality of Conjecture A in the function field case of characteristic p>0.
- Prove that the set of integral points on $X \setminus D$ is not Zariski-dense, in the number field case.
- Prove the inequality of Conjecture A in the number field case.
- Prove "moving targets" versions of the above.

For example, one may pose Conjecture A in each of the above contexts (in which case the first, fourth, sixth, eighth, ninth, and tenth entries would not apply, since they are false in some cases). If one restricted to pairs (X, D) of logarithmic general type $(i.e., \mathcal{X}_X(D))$ is big), then all of the above would apply. Or, one may pose other restrictions, as was done for Conjecture B.

The above hierarchy is ranked roughly from easiest to hardest. The general idea is that one would start from the top and work down from there, using the insight gained on earlier steps to help with the later ones. For example, this paper works through some of the above steps for Conjecture B. The first item is not useful for Conjecture B (since the exceptional set is all of X), the second item is solved in Section 2, the third in Section 4, the fourth item again is not useful, and the fifth item was solved in Section 5. The sixth item is not useful, so the next step is to try to prove Conjecture B in the function field case of characteristic zero.

As another example, McQuillan's paper [McQ 2] proved the Nevanlinna-theory analogue of Bogomolov's theorem bounding the number of curves of given genus on surfaces of general type with $c_1^2 > c_2$. This may be regarded as proceeding from the second step to the fourth.

§7. Complements

This section gives some variations of the method of Section 3. These give some interesting implications.

We begin with a variation that shows that Conjecture B, with D=0, would still imply a weak form of the abc conjecture, with (0.1.1) replaced by

(7.1)
$$\max\{|a|, |b|, |c|\} \le C \prod_{p|abc} p^{24+\epsilon}.$$

This shows that Conjecture B leads to nontrivial results even on rational varieties without a divisor. This does not augur well for the prospects of actually proving Conjecture B by known methods, since the absence of D and the triviality of the Albanese rule out the usual gains to be expected from taking a carefully chosen line sheaf on a product of several copies of the variety.

Proposition 7.2. There is a rational three-fold X with the property that if Conjecture B holds for X (with D=0) over \mathbb{Q} , then Conjecture 0.1 holds with (0.1.1) replaced by (7.1).

Proof. For relatively prime integers a, b, c with a + b + c = 0, pick integers u, v, w, x, y, and z such that x, y, and z are as large as possible, and such that

(7.2.1)
$$ux^5 = a, vy^5 = b, and wz^5 = c.$$

This defines a point

$$P_{a,b,c} = ([u:v:w], [x:y:z])$$

on the subvariety X of $\mathbb{P}^2 \times \mathbb{P}^2$ cut out by the equation

$$ux^5 + vy^5 + wz^5 = 0 .$$

This variety is smooth and rational, since the projection to the second factor exhibits it as a \mathbb{P}^1 -bundle over \mathbb{P}^2 . It admits an action of \mathbb{G}^2_m by

$$(g_1, g_2) \cdot ([u:v:w], [x:y:z]) = ([ug_1^{-5}:vg_2^{-5}:w], [xg_1:yg_2:z])$$
.

For i=1,2 let $\operatorname{pr}_i\colon X\to \mathbb{P}^2$ denote the projection to the i^{th} factor. Then, by the adjunction formula, $\mathscr{K}_X\cong\operatorname{pr}_1^*\mathscr{O}(-2)\otimes\operatorname{pr}_2^*\mathscr{O}(2)$. By functoriality of heights and by Conjecture B, it follows that

$$2h([x:y:z]) - 2h([u:v:w]) \le \epsilon(h([x:y:z]) + h([u:v:w])) + O(1)$$

outside a proper Zariski-closed subset $Z \subsetneq X$. Therefore,

$$(7.2.2) h([x:y:z]) \le h([u:v:w]) + \epsilon h([a:b:c]) + O(1)$$

(with a different ϵ). Thus, by (7.2.1) and (7.2.2),

$$\begin{split} h([a:b:c]) &\leq h([u:v:w]) + 5h([x:y:z]) \\ &\leq 6h([u:v:w]) + 5\epsilon h([a:b:c]) + O(1) \\ &\leq 24n(abc) + 5\epsilon h([a:b:c]) + O(1) \;. \end{split}$$

Here the last step follows because any given prime may occur in uvw to at most a fourth power. This gives (7.1) (with a different ϵ).

As in Section 3, the set Z corresponds to at most finitely many triples a,b,c, and so it can be ignored.

Remark 7.3. The above approach will also work if the exponents are changed from 5 to 4; this gives $27 + \epsilon$ instead of $24 + \epsilon$.

Remark 7.4. The above variety X (modified as in Remark 7.3) also appears in conjunction with the abc conjecture in the paper [McQ 1], where it is noted that a different conjecture also would imply the abc conjecture.

We mention here one more example of this sort, involving in this case a surface that can be explicitly described. We consider Pell's equation

$$x^2 - dy^2 = \pm 4, \qquad d \in \mathbb{Z} .$$

This equation potentially infinitely many solutions $(x, y) \in \mathbb{Z}^2$.

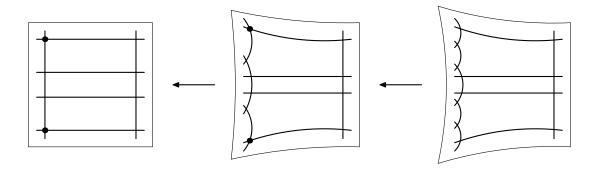
We consider here the question of whether these solutions are **mostly square free**; *i.e.*, whether for all fixed $\epsilon > 0$, the largest square factor of x or y is $O(\max\{|x|,|y|\})$. If the abc conjecture is true, then an easy argument implies that such solutions are mostly square free. Although the converse does not seem to hold, the question of whether solutions to Pell's equation are mostly square free still captures some of the flavor of the abc conjecture.

Proposition 7.5. There is a rational projective surface X and a divisor D on X such that (X, D) is regular, and such that if Conjecture B holds for (X, D) over all quadratic fields, then solutions of Pell's equation are mostly square free.

Proof. This follows by applying essentially the same methods as in Section 3 and the earlier part of this section, applied to the equation

$$x^2v^4 - y^2w^4 = \pm 4$$
,

which determines a variety in $\mathbb{P}^1 \times \mathbb{P}^2$ whose desingularization is easy to find explicitly. The resulting variety X can be described pictorially below.



Here the diagram on the left depicts the divisor

$$\operatorname{pr}_{1}^{*}([0] + [\infty] + [1] + [-1]) + \operatorname{pr}_{2}^{*}([0] + [\infty])$$

in $\mathbb{P}^1 \times \mathbb{P}^1$, with the lines $\operatorname{pr}_1^{-1}(\pm 1)$ drawn in the middle. The arrows are blowings-up at the fat points. Since $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at one point is isomorphic to the blowing-up of \mathbb{P}^2 at two points, X can also be described as a certain blowing-up of \mathbb{P}^2 , with D being the inverse image of the coordinate axes and the lines $y=\pm x$.

We leave the details of this proof to the reader.

Although the surface X is just a few blowings-up away from the more general version of Schmidt's Subspace Theorem ([Vo 1], Thm. 2.5.8), it is still, unfortunately, out of reach.

Appendix A. Proof of Proposition 5.1

This appendix provides a proof of Proposition 5.1, since a proof has not yet appeared in print. The proof presented here relies on a geometric formulation of the classical "lemma on the logarithmic derivative"; see Theorem A.2. This may be of independent interest.

This geometric logarithmic derivative lemma was first proved by M. McQuillan [McQ 2], but was stated only implicitly. See also [McQ 3]. Theorem A.6 is also due to him; the proof given here is based on his ideas but uses more elementary methods. The proof of Theorem A.2 presented here does not follow McQuillan; rather, it reduces to the classical logarithmic derivative lemma by a method originally due to P.-M. Wong [W]. In ([W], Thm. 4.1), Wong proved the following special case: If X is a smooth projective variety, if D is an effective divisor with strict normal crossings (Definition A.2.3), if $f: \mathbb{C} \to X \setminus D$ is a non-constant holomorphic map, if ω is a global section of $\int_k^m X(\log D)$, and if $(j^k f)^* \omega$ is not identically zero, then

$$\int_0^{2\pi} \log^+ \left| \omega(j^k f(re^{i\theta})) \right| \frac{d\theta}{2\pi} \le_{\text{exc}} O(\log T_{\mathscr{A},f}(r)) + O(\log r) .$$

We begin with the geometric logarithmic derivative lemma. Let X be a smooth compact complex algebraic variety, let D be a normal crossings divisor on X, and let $f\colon X\to\mathbb{C}$ be a holomorphic curve whose image is not contained in the support of D. Let $V=\mathbb{V}(\Omega^1_X(\log D))$ and $\overline{V}=\mathbb{P}(\Omega^1_X(\log D)\oplus \mathscr{O}_X)$. Here we use the conventions of ([**EGA II**], 1.7.8), so that V is the total space of the tangent sheaf of X with logarithmic zeroes along D and \overline{V} is the obvious projective completion. The natural map $f^*\colon \Omega^1_X(\log D)\to \Omega^1_\mathbb{C}(\log f^*D)$, together with the map $\mathscr{O}_X\to \Omega^1_\mathbb{C}$ given by $1\mapsto dz$, gives a holomorphic map $Df\colon \mathbb{C}\to \mathbb{P}(\Omega^1_X(\log D)\oplus \mathscr{O}_X)$. (Note that this differs from the map $f'\colon \mathbb{C}\to \mathbb{P}(\Omega^1_X(\log D))$ defined in Section 5.) Let $[\infty]$ be the complement of V in \overline{V} , and choose a Weil function (or Green function) $g_{[\infty]}$ for this divisor. We will use the normalization of (A.1), below.

One possible way to choose this Weil function is the following. Choose a Hermitian metric on $\Omega_X^1(\log D)$; its dual metric on the logarithmic tangent bundle $T_X(-\log D)$ induces a Weil function for $[\infty]$ by the formula

(A.1)
$$g_{[\infty]}(P) = \log^+ \|\xi\|$$
,

where $P \in V$ corresponds to an element ξ in the fiber of $T_X(-\log D)$ over the corresponding point of X, and $\log^+ x$ is defined as $\max\{\log x, 0\}$.

The geometric logarithmic derivative lemma can then be stated as follows.

Theorem A.2. Let (X, D) be a regular pair with X complete over \mathbb{C} , let $f: \mathbb{C} \to X$ be a holomorphic curve whose image is not contained in the support of D, and let \mathscr{A} be a line sheaf on X whose restriction to the Zariski closure of the image of f is big. Let Df and $m_{Df}([\infty], r)$ be as above. Then

(A.2.1)
$$m_{Df}([\infty], r) \leq_{\text{exc}} O(\log^+ T_{\mathscr{A}, f}(r)) + o(\log r) .$$

Proof. The proof works by reducing to a situation where the classical lemma on the logarithmic derivative can be applied a finite number of times. We begin with some reductions.

Lemma A.2.2. Let (X_2, D_2) be a regular pair with X_2 complete over \mathbb{C} , and let $\pi \colon X_2 \to X$ be a morphism that induces a birational morphism $X_2 \to \pi(X_2)$. Assume also that $\operatorname{Supp} D_2 = \pi^{-1}(\operatorname{Supp} D)$ and that the image of f is a Zariskidense subset of $\pi(X_2)$. Let $g \colon \mathbb{C} \to X_2$ be the (unique) holomorphic curve such that $\pi \circ g = f$. Let \mathscr{A}_2 be a big line sheaf on X_2 . Finally, assume that Theorem A.2 holds for g and \mathscr{A}_2 . Then it also holds for f and \mathscr{A} .

Proof. Let $V_2 = \mathbb{V}(\Omega^1_{X_2}(\log D_2))$ and let $\overline{V}_2 = \mathbb{P}(\Omega^1_{X_2}(\log D_2) \oplus \mathscr{O}_{X_2})$. The natural map $\pi^*\Omega^1_X \to \Omega^1_{X_2}$ induced by π extends to a map $\pi^*\Omega^1_X(\log D) \to \Omega^1_{X_2}(\log D_2)$. This defines a morphism $V_2 \to V \times_X X_2$ and hence a morphism $D\pi \colon V_2 \to V$. Moreover, $Df|_{(Dg)^{-1}(\operatorname{Supp} D_2)} = D\pi \circ Dg|_{(Dg)^{-1}(\operatorname{Supp} D_2)}$. We claim that

(A.2.2.1)
$$m_{Df}([\infty], r) \le m_{Dg}([\infty], r) + O(1)$$
.

Indeed, we may assume that the Weil functions for the respective divisors $[\infty]$ on \overline{V} and \overline{V}_2 are of the form (A.1). The inequality then follows from the fact that the natural map $\theta \colon T_{X_2}(-\log D_2) \to \pi^*T_X(-\log D)$ satisfies $\|\theta(\xi)\| \ll \|\xi\|$ for all $\xi \in T_{X_2}(-\log D_2)$ by compactness of X_2 .

To compare the error terms, we have

(A.2.2.2)
$$T_{\mathcal{A},f}(r) \gg T_{\mathcal{A}_2,g}(r) + O(1)$$

since $\pi^* \mathscr{A}$ is big. The lemma then follows from (A.2.2.1) and (A.2.2.2).

Let Z be the Zariski closure of the image of f. By letting $\pi: X_2 \to X$ be a resolution of the pair $(Z, D|_Z)$, we may assume that the image of f is Zariski dense.

By applying Chow's lemma and resolving singularities again, we may further assume that X is projective.

Definition A.2.3. A divisor D on a smooth variety X has **strict normal crossings** (also called **simple normal crossings**) if it is a normal crossings divisor and all of its irreducible components are regular.

It is well known that in the present situation there exists a smooth projective variety X_2 and a birational morphism $\pi\colon X_2\to X$ such that $\pi^{-1}(\operatorname{Supp} D)$ is the support of a divisor D_2 on X_2 with strict normal crossings, and such that π is an isomorphism outside of the support of D_2 . After applying Lemma A.2.2 to π , we may therefore assume that D has strict normal crossings.

The remainder of the proof follows the method of P.-M. Wong $[\mathbf{W}]$:

Let $n = \dim X$, let \mathcal{L}_0 be a very ample line sheaf on X, let $E_{00}, \ldots, E_{0,2n}$ be effective divisors corresponding to \mathcal{L}_0 such that any n+1 have empty intersection,

and for integers i, j with $0 \le i < j \le 2n$ choose a rational function f_{ij} on X such that $(f_{ij}) = E_{0i} - E_{0j}$. Then the set

$$\left\{ \frac{df_{ij}}{f_{ij}} : 0 \le i < j \le 2n \right\}$$

of rational sections of Ω^1_X has the property that, for each point $P \in X$, some subset of this set is regular at P and generates Ω^1_X there.

Next, recalling that D has strict normal crossings, write $D = D_1 + \cdots + D_\ell$, where each D_i is effective and has smooth support. For each $i = 1, \ldots, \ell$ let \mathcal{L}_i and \mathcal{L}'_i be very ample line sheaves on X such that $\mathcal{L}_i \cong \mathcal{L}'_i(D_i)$. For each i and each $j = 1, \ldots, n$ choose effective divisors E_{ij} and E'_{ij} associated to \mathcal{L}_i and \mathcal{L}'_i , respectively, such that $\bigcap_j (\operatorname{Supp} E_{ij} \cup \operatorname{Supp} E'_{ij})$ is disjoint from $\operatorname{Supp} D_i$. For each i and j choose a rational function g_{ij} on X such that $(g_{ij}) = D_i + E'_{ij} - E_{ij}$. Then the set

$$\left\{ \frac{dg_{ij}}{g_{ij}} : 1 \le i \le \ell, \ 1 \le j \le n \right\}$$

of rational sections of $\Omega^1_X(\log D)$ has the property that, for each point $P \in X$, some subset of this set is regular at P and generates $\Omega^1_X(\log D)$ over Ω^1_X there.

Let

$$\mathcal{H} = \{ f_{ij} : 0 \le i < j \le 2n \} \cup \{ g_{ij} : 1 \le i \le \ell, \ 1 \le j \le n \} .$$

This set has the property that, for each $P \in X$, there is a subset \mathscr{H}_P of \mathscr{H} such that dh/h is a regular section of $\Omega^1_X(\log D)$ at P for all $h \in \mathscr{H}_P$, and these sections generate $\Omega^1_X(\log D)$ there. By compactness of X, it then follows that

$$g_{[\infty]}(Df(z)) \le \log^+ \max_{h \in \mathscr{H}} \left| \frac{(h \circ f)'}{h \circ f} \right| + O(1)$$
.

Thus, by the classical lemma on the logarithmic derivative applied to the meromorphic functions $h \circ f$, $h \in \mathcal{H}$, we have

$$m_{Df}([\infty], r) \leq \max_{h \in \mathscr{H}} \int_{0}^{2\pi} \log^{+} \left| \frac{(h \circ f)'(re^{i\theta})}{(h \circ f)(re^{i\theta})} \right| \frac{d\theta}{2\pi} + O(1)$$

$$\leq_{\text{exc}} \sum_{h \in \mathscr{H}} O(\log^{+} T_{h \circ f}(r)) + o(\log r)$$

$$\leq_{\text{exc}} O(\log^{+} T_{\mathscr{A}, f}(r)) + o(\log r) .$$

Remark A.3. When $X = \mathbb{P}^1$ and $D = [0] + [\infty]$, this theorem reduces to the classical lemma on the logarithmic derivative. Indeed, in that case $T_X(-\log D)$ is the trivial line bundle on \mathbb{P}^1 , via the isomorphism $z\frac{\partial}{\partial z}\mapsto 1$. Let $f\colon\mathbb{C}\to\mathbb{P}^1$ be a holomorphic map whose image is not contained in $\{0,\infty\}$. Then $g_{[\infty]}(Df(z)) = \log^+|f'(z)/f(z)| + O(1)$, so in this case

$$m_{Df}([\infty], r) = \int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| \frac{d\theta}{2\pi} + O(1) ,$$

which is the quantity appearing in the classical lemma on the logarithmic derivative.

Remark A.4. Since the above proof merely reduces to multiple applications of the classical case, any sharpening of the error term in the classical lemma on the logarithmic derivative leads immediately to a correspondingly sharp error term in the above generalization (up to a constant factor). Here we used the error term of ([S], Thm. 3.11) or ([Y], Thm. 1).

Proposition 5.1 is a fairly easy consequence of the above geometric lemma on the logarithmic derivative. It will be proved in a slightly stronger form, involving a modified counting function for ramification. Recall that $\overline{V} = \mathbb{P}(\Omega_X^1(\log D) \oplus \mathscr{O}_X)$, and let $p \colon P \to \overline{V}$ be the blowing-up of \overline{V} along the zero section (of V); i.e., the section corresponding to the projection $\Omega_X^1(\log D) \oplus \mathscr{O}_X \to \mathscr{O}_X$. Let $[0] \subseteq P$ be the exceptional divisor and let $\phi \colon \mathbb{C} \to P$ be the lifting of $Df \colon \mathbb{C} \to \overline{V}$.

Definition A.5. The *D*-modified ramification counting function of a holomorphic curve $f: \mathbb{C} \to X$ is the counting function for $\phi^*[0]$:

$$N_{\text{Ram}(D),f}(r) = N_{\phi}([0],r)$$
.

Theorem A.6. Let (X,D) be a regular pair with X complete over \mathbb{C} , let $f:\mathbb{C} \to X$ be a holomorphic curve not lying entirely within the support of D, and let \mathscr{A} be a line sheaf on X whose restriction to the Zariski closure of the image of f is big. Let $f':\mathbb{C} \to \mathbb{P}(\Omega^1_X(\log D))$ be the canonical lifting of f, and let $\mathscr{O}(1)$ denote the tautological line sheaf on $\mathbb{P}(\Omega^1_X(\log D))$. Then

(A.6.1)
$$T_{\mathcal{O}(1),f'}(r) \leq_{\text{exc}} N_f^{(1)}(D,r) - N_{\text{Ram}(D),f}(r) + O(\log^+ T_{\mathcal{A},f}(r)) + o(\log r)$$
.

Proof. This proof essentially follows McQuillan [McQ 2], but some details are different. Recall the blowing-up $p: P \to \mathbb{P}(\Omega^1_X(\log D) \oplus \mathscr{O}_X)$. This P admits a morphism $q: P \to \mathbb{P}(\Omega^1_X(\log D))$, extending the rational map

$$\mathbb{P}(\Omega^1_X(\log D) \oplus \mathscr{O}_X) \dashrightarrow \mathbb{P}(\Omega^1_X(\log D))$$

associated to the canonical map $\Omega^1_X(\log D) \hookrightarrow \Omega^1_X(\log D) \oplus \mathscr{O}_X$. We first compare the pullbacks $q^*\mathscr{O}(1)$ and $p^*\mathscr{O}(1)$ of the tautological line sheaves on $\mathbb{P}(\Omega^1_X(\log D))$ and $\overline{V} = \mathbb{P}(\Omega^1_X(\log D) \oplus \mathscr{O}_X)$, respectively. Let s be any nonzero rational section of $\Omega^1_X(\log D)$. This determines a rational section of $\mathscr{O}(1)$ on $\mathbb{P}(\Omega^1_X(\log D))$. The corresponding divisor D_1 is the sum of a component which is generically a hyperplane section on fibers over X, and the pull-back of a divisor on X. But also (s,0) is a nonzero rational section of $\Omega^1_X(\log D) \oplus \mathscr{O}_X$, giving a rational section of $\mathscr{O}(1)$ on \overline{V} , hence a divisor D_2 which is again the sum of a generic hyperplane section and the pull-back of a divisor on X. Comparing q^*D_1 with p^*D_2 , we see that they coincide except that p^*D_2 contains [0] with multiplicity 1. Hence

$$q^* \mathcal{O}(1) \cong p^* \mathcal{O}(1) \otimes \mathcal{O}(-[0])$$
.

The global section (0,1) of $\Omega_X^1(\log D) \oplus \mathscr{O}_X$ corresponds to the divisor $[\infty]$ on \overline{V} ; hence $\mathscr{O}([\infty]) \cong \mathscr{O}(1)$ and therefore

$$(A.6.2) T_{\mathscr{O}(1),f'}(r) = N_{Df}([\infty],r) + m_{Df}([\infty],r) - N_{\phi}([0],r) - m_{\phi}([0],r) + O(1) .$$

The lifted curve Df meets $[\infty]$ only over D, and with multiplicity at most 1. Hence

(A.6.3)
$$N_{Df}([\infty], r) \le N_f^{(1)}(D, r)$$
.

The second term $m_{Df}([\infty], r)$ is bounded by Theorem A.2. The third term (by definition) is equal to the modified ramified counting function, and the fourth term on the right-hand side of (A.6.2) is bounded from below and therefore can be ignored. Thus (A.6.2) gives (A.6.1).

Finally, we note that Theorem A.6 implies Proposition 5.1 since $N_{\text{Ram}(D),f}(r) \geq 0$. Again, sharper error terms in the classical lemma on the logarithmic derivative lead to sharper error terms in Theorem A.6.

Theorem A.6 can also be proved in the context of coverings (classically called "algebroid functions"). In this context, one has a finite ramified covering $p: Y \to \mathbb{C}$ and a holomorphic function $f: Y \to X$. In place of the set $\{z \in \mathbb{C} : |z| = r\}$, let $Y\langle r \rangle = \{z \in Y : |p(z)| = r\}$. The classical lemma on the logarithmic derivative, originally due to Valiron $[\mathbf{Va}]$, can then be stated as follows.

Theorem A.7. Let Y, p, and f be as above. Then, for all $\epsilon > 0$,

$$\int_{Y(r)} \log^{+} \left| \frac{1}{f(z)} \frac{df}{dp}(z) \right| d^{c} \log |p(z)|^{2} \leq_{\text{exc}} (\deg p + \epsilon) [7 \log T_{f}(r) + 6 \log^{+} r] .$$

Proof. This follows from ([**A**], Thm. 2.2) with $\tau = |p|^2$, $\mathbb{B} = 1$, $\Theta = dp$, and other notation as in ([**A**], § 1.1).

To state Theorem A.2 in the context of coverings, we define Df to be the map $Y \to \overline{V}$ associated to the map $f^* \colon \Omega^1_X(\log D) \to \Omega^1_Y(\log f^*D)$, together with the map $\mathscr{O}_X \to \Omega^1_Y$ given by $1 \mapsto dp$. The statement and proof of Theorem A.2 then carry over directly, with (A.2.1) replaced by

(A.8)
$$\int_{Y\langle r\rangle} g_{[\infty]}(Df(z))d^c \log |p(z)|^2 \leq_{\text{exc}} O(\log T_{\mathscr{A},f}(r)) + O(\log^+ r) .$$

Then Theorem A.6 holds with (A.6.1) replaced by

(A.9)
$$T_{\mathcal{O}(1),f'}(r) \leq_{\text{exc}} N_f^{(1)}(D,r) + N_{\text{Ram},p}(r) - N_{\text{Ram}(D),f}(r) + O(\log T_{\mathcal{A},f}(r)) + O(\log^+ r) ,$$

where the additional term $N_{\text{Ram},p}(r)$ is the counting function for the ramification of $p: Y \to \mathbb{C}$. The proof is again essentially the same, except that the additional term $N_{\text{Ram},p}(r)$ needs to be added to the right-hand side of (A.6.3).

As has been noted by R. Kobayashi, the above results also hold for higher jet bundles. This can be seen by noting that the higher jet bundle is the tangent bundle of the next lower jet bundle. The details are left to the reader.

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